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On the topology of vortex lines and tubes

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This paper examines the widespread idea that vortex lines and tubes must either close on themselves or extend to the boundary of the fluid. A survey of the origins of this misconception, and of earlier attempts to set it right, is followed by an analysis of simple flows exhibiting vortex lines and tubes which do not fit those shapes. Two types of vortex lines are discussed: *dense*, which comprise open lines of infinite length but confined in a finite region, and *separatrix*, which comprise lines that begin or finish within the fluid, at points where the vorticity is null. The presence of these vortex lines in a vortex tube affects its topology in the following ways. Vortex tubes formed by dense vortex lines have infinite length; they self-intersect an infinite number of times but do not close on themselves. Vortex tubes formed by separatrix vortex lines (and either closed or open vortex lines) are torn apart at the points where the vorticity is null. Vortex tubes exclusively composed of separatrix vortex lines begin or finish at points or surfaces within the fluid; in this particular situation the vortex tube has zero strength.

1. Introduction

Almost 150 years ago Helmholtz (1858) published a paper, Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen, that launched the study of vortex dynamics and exercised a major influence on other areas of physics and mathematics (see e.g. Epple 1998; Darrigol 2005). In this epoch-making paper Helmholtz studied the motion of an ideal fluid subjected to conservative forces but he did not assume that the velocity is the gradient of a potential function, as was common at the time. He began by showing that the motion of an infinitesimal volume of a continuous medium can be decomposed into a translation, compression or expansion in three mutually orthogonal directions, and a rotation. For a more intuitive understanding of his results, Helmholtz introduced two new concepts: *vortex line*, which is a line that at all its points has the direction of the instantaneous vorticity of fluid particles; and *vortex filament*, which is a portion of fluid limited by the vortex lines that pass through the perimeter of an infinitesimal element of surface. With these definitions, his main results can be stated as follows: (1) fluid particles originally free of vorticity remain free of vorticity, (2) fluid particles which at any time form a vortex line, however they move, continually form a vortex line, and (3) the product of the cross-section and the vorticity of a vortex filament is constant on the whole length of the filament and does not change in time. The constancy of the flux of vorticity along the vortex filament is a kinematic theorem: Helmholtz proved it using solely the divergenceless character of the vorticity field. From this theorem Helmholtz derived an incorrect corollary: vortex filaments must close on themselves or extend to the boundary of the fluid. He substantiated his conclusion as follows: if the vortex filament ended somewhere within the fluid then it would be possible to construct a closed surface through which the flux of vorticity would not be zero (which is impossible by virtue of the divergence theorem and the identity $\nabla \cdot \boldsymbol{\omega} = 0$). This argument, however, is incomplete and does not prove the corollary (Chorin & Marsden 1993).

The simple shapes and invariant strengths of Helmholtz's vortices captured the imagination of Thomson (1867), who hypothesized that atoms were vortices in an all-pervading, ideal fluid. In order to advance his conjecture, which turned out to be erroneous, Thomson (1869) further developed the mathematical theory of vortex motion. He established the circulation theorem that now bears his name† and showed that Helmholtz's two theorems on vortex filaments, namely that vorticity flux is uniform along the filament and that it is constant in time, also hold for a vortex tube, which he defined as a surface formed by all the vortex lines that pass through a closed contour. Surprisingly, he derived an analogous corollary: vortex tubes must close on themselves or extend to the boundary of the fluid. Maxwell (1875) arrived at the same conclusion in his essay published in the ninth edition of the *Encyclopædia Britannica*, and a few years later Lamb (1879) derived a similar result for vortex lines.

When misconceptions have such illustrious origins it is hardly surprising that they spread widely. Thus, most classic textbooks on fluid mechanics contain incorrect assertions about vortex lines, filaments or tubes (Lamb 1932; Goldstein 1960; Sommerfeld 1950; Lighthill 1963; Batchelor 1967). And although some of the errors have been exposed from time to time (Hadamard 1903; Truesdell 1954; Chorin & Marsden 1993) they continue to appear in modern texts (see e.g. Saffman 1995; Kundu & Cohen 2002; Cottet & Koumoutsakos 2004) and in research papers published in leading scientific journals (see e.g. Widnall 1975; Saffman 1990; Ottino 1990; Robinson 1991; Sarpkaya 1996; Webster & Longmire 1998; Zhang, Shen & Yue 1999; Nolan 2001; Dickinson 2003; Chadwick 2005; Barranco & Marcus 2005).

The constancy of the flux along vortex tubes depends only on the vorticity field being divergenceless. Therefore, vector tubes of any vector field with this property have a constant flux too. Oddly, the erroneous assertion that vector lines or vector tubes must close on themselves or extend to the boundary has also been made about other divergenceless vector fields (Feynman, Leighton & Sands 1964; Fetter 1967; Batchelor 1967, Cingoski *et al.* 1996).

The object of the present study is to show, through analytic examples, that the possible shapes of vortex lines (§ 2) and vortex tubes (§ 3) is more diverse and intricate than generally believed.

2. Vortex lines

Vortex lines are, by definition, the solution curves of

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \boldsymbol{\omega}(\boldsymbol{x}), \tag{2.1}$$

where $\boldsymbol{\omega}$ is the vorticity field, $\boldsymbol{x}$ denotes the position in three-dimensional space, and s is a parameter. If $\boldsymbol{x} = \boldsymbol{X}$ when $s = 0$, the solution is $\boldsymbol{x} = \boldsymbol{\Phi}_s(\boldsymbol{X})$, where $\boldsymbol{X} = \boldsymbol{\Phi}_{s=0}(\boldsymbol{X})$, i.e. a point traces a curve from $\boldsymbol{X}$ to $\boldsymbol{x}$ as the independent variable changes from 0 to s. Mathematically, (2.1) defines a dynamical system and the solution $\boldsymbol{\Phi}_s$ is called *flow* or

† William Thomson became Baron Kelvin of Largs in 1892; Hermann Helmholtz was ennobled in 1882 and added von to his name. We will, however, use their names as they appeared on the papers discussed here.

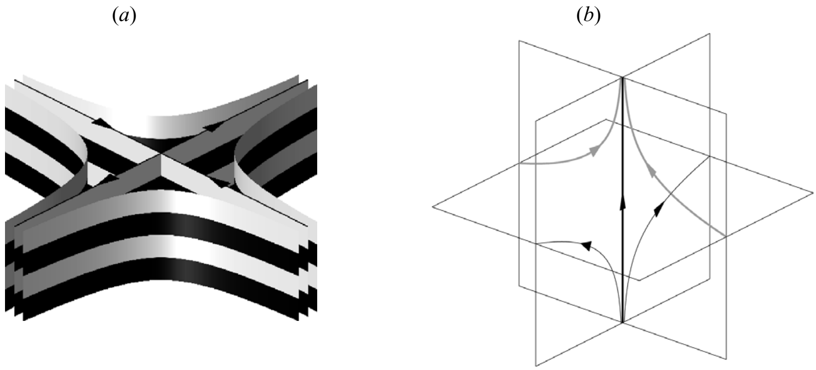

FIGURE 1. (*a*) Vortex surfaces of the vorticity field $\boldsymbol{\omega}=(x,-y,0)$; the boundaries between black and grey stripes are vortex lines; the black lines and arrows on the top edges of the flat surfaces represent, respectively, vortex lines and the direction of the vorticity vector. (*b*) These straight vortex lines appear to cross but, when represented in the space (x, y, s), turn out to be five solution curves of equation (2.1) that do not cross (see text).

motion (these are not to be confused with the actual flow or motion of fluid particles). The solution is unique if there is a constant L such that, in an interval s, the flow $\boldsymbol{\Phi}_s$ produced by $\boldsymbol{\omega}$ increases the distance between two arbitrary points no more than L times (Lipschitz condition, see e.g. Arnold 1973). In this case only one vortex line can pass through each point and thus vortex lines cannot intersect. Moreover, since $\nabla \cdot \boldsymbol{\omega} = 0$, the *flow* $\boldsymbol{\Phi}_s$ preserves both volume and topology. The former means that $\boldsymbol{\Phi}_s$ maps each region of space to another region of equal volume; the latter means that any region is mapped to a topologically equivalent region (e.g. a closed curve is mapped to a closed curve and not to, say, an open curve or two closed curves). In the analysis of the topology of vortex lines and tubes no reference will be made to the dynamical equations that govern the motion of the fluid; therefore our results are valid for vector lines and tubes of any divergenceless vector field, such as the velocity field of an incompressible fluid (the continuity equation is then $\nabla \cdot \boldsymbol{u} = 0$) or the magnetic field $\boldsymbol{B}$ (one of Maxwell's equations is precisely $\nabla \cdot \boldsymbol{B} = 0$).

Vortex lines can begin or end within the fluid at points where $\boldsymbol{\omega}=0$ (called equilibrium or fixed points of equation (2.1)). Some authors consider that the continuity of vector lines beyond fixed points is a matter of convention (Kellogg 1929; Saffman 1995), but assuming that vector lines continue beyond these points usually contradicts the theorem of uniqueness of solutions. Consider, for instance, the parallel flow given by $\boldsymbol{u}=(0,0,xy)$ which has the associated vorticity field $\boldsymbol{\omega}=(x,-y,0)$. Every point of the z-axis is an equilibrium point of the vorticity field (i.e. $\boldsymbol{\omega}(0,0,z)=0$) and the vortex surfaces in the neighbourhood of this line are as illustrated in figure 1(*a*). Apparently these surfaces (and the vortex lines they are made of) intersect transversally on the z-axis. This, however, is not possible because $\boldsymbol{\omega}$ satisfies the Lipschitz condition and hence only one vortex line passes through any point. We can see what is happening around the equilibrium points by using a different representation of the vortex lines. Since these are solution curves of equation (2.1), they can also be represented in the extended space (x, y, z, s). Note, however, that in our example the vorticity is two-dimensional: the component in the z-direction is identically zero and the other components do not depend on z. We can thus drop the z coordinate and use the space (x, y, s). Figure 1(*b*) represents in this space the vortex lines that appear to cross when viewed in the physical space (x, y, z). The two straight vortex lines are in fact five solutions: the equilibrium solution (represented by a black thick line), two solutions that asymptotically approach the equilibrium point as $s \to \infty$ (grey thick lines) and two solutions that asymptotically approach the equilibrium point as $s \to -\infty$ (black thin lines). The solutions that approach the equilibrium point correspond to vortex lines that end within the fluid; the solutions that move away correspond to vortex lines that begin within the fluid. By analogy with dynamical systems these may be called separatrix vortex lines.

It has long been known that the simple, closed loops of vorticity referred to in textbooks are exceptional. In general, a vortex line has infinite length and passes infinitely often infinitely close to itself (Hadamard 1903; Truesdell 1954; Moffatt 1969; Saffman 1995). This is a consequence of the recurrence theorem of Poincaré (1890) which can be paraphrased as follows: if a flow has only bounded vortex lines, then for any volume, however small, there exist vortex lines that intersect the volume an infinite number of times. Note that a closed vortex line, being a periodic solution of equation (2.1), is understood to pass infinitely often through each of its points.

A case in point is that of spherical and ring vortices with swirl (Moffatt 1969, 1988). In these flows the vorticity can be written, using cylindrical coordinates (r, θ, z), as follows:

$$\boldsymbol{\omega} = \left(-\frac{1}{r}\frac{\partial \Psi}{\partial z}, \omega_\theta, \frac{1}{r}\frac{\partial \Psi}{\partial r} \right), \tag{2.2}$$

where $\Psi = \Psi(r, z)$ is the vorticity function and $\omega_\theta = \omega_\theta(r, z)$ is the azimuthal vorticity. Vortex lines lie on doughnut-shaped surfaces where the vorticity function Ψ has a constant value, say p. Vortex lines coil around these surfaces, so they are described by the equations

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \Omega_\theta(p), \qquad \frac{\mathrm{d}\lambda}{\mathrm{d}s} = \Omega_\lambda(p), \tag{2.3}$$

where θ is the angle around the symmetry axis (i.e. it is the same angle as the cylindrical coordinates), λ is the angle around the centreline of the torus, and Ω_θ and Ω_λ are the corresponding rotation numbers. If the ratio $\Omega_\theta/\Omega_\lambda$ is a rational number the vortex line closes on itself; if it is irrational the vortex line winds around the torus without ever closing on itself, thus densely filling the surface. Since rational numbers are the exception and irrational numbers the rule, only in exceptional cases do vortex lines form closed loops.

Something analogous happens with streamlines in Taylor vortices, which occur in the fluid contained between two coaxial cylinders rotating at different speeds. Each vortex has the shape of a doughnut with a squared cross-section (Taylor 1923, see his figure 5) and fluid particles rotate both around the axis and around the vortex's centreline. When the ratio between the two rotation periods is a rational number the streamline closes on itself; when it is irrational the streamline has infinite length and densely fills a surface. A more complicated situation occurs in the quadratic flow inside a sphere studied by Bajer & Moffatt (1999). This is a steady Stokes flow with chaotic streamlines in at least part of the flow domain; thus streamlines in these regions densely fill a finite volume. Of course, in these examples it is the streamlines rather than the vortex lines that are dense but, since the two velocity fields are divergenceless, one could conceive of vorticity fields with corresponding topologies. In point of fact, Moffatt (1969) suggests that vortex lines in a turbulent blob are generally dense.

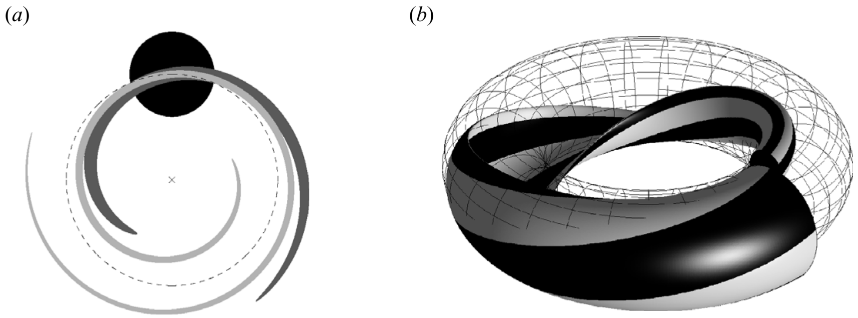

FIGURE 2. Vortex tubes of infinite length in a vortex ring with swirl. (*a*) Successive intersections of a tube with a meridional plane shows that the tube intersects with itself but never closes. (*b*) A three-dimensional view of a segment of this tube (the boundaries between black and grey stripes are vortex lines).

3. Vortex tubes

Hadamard (1903) noted that the existence of dense vortex lines has consequences for the topology of vortex tubes, but he was too concise and wrote only that "similar observations apply to vortex tubes". Others have failed to mention these consequences and, along with statements about the existence of dense vortex lines, still include the assertion that vortex tubes must be closed or extend to the boundary (see e.g. Truesdell 1954; Chorin & Marsden 1993; Saffman 1995).

So let us analyse the topology of a vortex tube in spherical and ring vortices with swirl. As discussed above, in these flows vortex lines are dense or, exceptionally, form closed loops. Depending on the location of the contour C used to generate the vortex tube, this may be formed by (1) closed vortex lines only, (2) dense vortex lines only, or (3) a combination of both closed and dense vortex lines. Instances (1) and (2) occur when C encircles one of the doughnut-shaped surfaces $\Psi = \text{const.}$; then if the rotation number of this surface is rational (1) occurs, and if it is irrational (2) occurs. In both cases the vortex tube is identical to the doughnut-shaped surface $\Psi = \text{const.}$, so it is a closed surface. Instance (3) occurs when C intersects different doughnut-shaped surfaces $\Psi = \text{const.}$; in this case the tube has infinite length and never closes. At first sight this situation seems conflicting, if not impossible: a finite cross-section multiplied by an infinite length would result in an infinite volume. A vortex tube, however, may intersect with itself without being closed: a tube of infinite length may thus exist within a finite region of space. As an example, consider the vortex tube formed by the vortex lines passing through C, which is a small circle on a meridional plane (e.g. the perimeter of the black circle in figure 2*a*). The ratio $\Omega_\theta/\Omega_\lambda$ varies continuously across the tori intersected by C, therefore through C pass dense vortex lines as well as closed vortex lines. Figure 2(*a*) shows the first three intersections of the tube with the meridional plane (the initial one is shown in black and subsequent intersections are shown in grey). In this particular example the cross-section has constant area but its perimeter increases continuously. It is clear that the vortex tube will intersect endlessly with itself but it will never close. Figure 2(*b*) shows the first turn that the tube makes around the symmetry axis.

Similarly, Taylor vortices display stream tubes that self-intersect an infinite number of times without closing on themselves. This is a counter-example to the statement that in an incompressible fluid a stream tube must either be closed or extend to the boundary (Batchelor 1967).

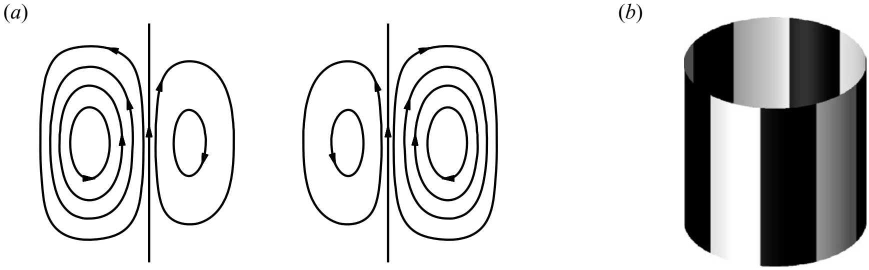

FIGURE 3. (*a*) The vorticity field given by equation (3.2) represented in a meridional plane: all vortex lines are closed, except for the separatrix vortex lines. (*b*) The vortex tube formed by all such separatrix vortex lines has zero strength and begins and ends on the planes $z=-1, 1$, respectively.

The vorticity flux ($\int \boldsymbol{\omega}\cdot \mathrm{d}s$) out of any reducible closed surface is zero by virtue of the divergence theorem and the identity $\nabla\cdot\boldsymbol{\omega}=0$. This property is invoked when arguing that a vortex tube cannot end within the fluid. For then it would be possible to find a closed surface out of which the flow of vorticity would be non-zero. The argument assumes that the strength of the vortex is finite, but if it is zero the tube can end at a surface within the fluid without contradicting the solenoidal character of the vorticity field (this has been recognized by Chorin & Marsden 1993). Consider the following example: a fluid of infinite extent which is everywhere at rest except in the disk $r<2$, $-1<z<1$, where the velocity and vorticity fields are given, respectively, by

$$\boldsymbol{u} = f(r)g(z)\boldsymbol{\theta}, \tag{3.1}$$

$$\boldsymbol{\omega} = -f(r)g'(z)\boldsymbol{r} + \left(f'(r) + \frac{f(r)}{r}\right)g(z)\boldsymbol{k}, \tag{3.2}$$

with $f(r)=(r-1)^5-2(r-1)^3+(r-1)$ and $g(z)=z^4-2z^2+1$. This is an axially symmetric flow consisting of two adjacent jets circling in opposite directions. The geometry of the vortex lines in a meridional plane is shown in figure 3(*a*). In this plane all vortex lines are closed except for the two straight lines, which begin and end within the fluid. If we construct a vortex tube by taking all the vortex lines that pass through the circle $r=1$, $z=0$ we obtain a cylindrical vortex tube of zero strength that begins on the plane $z=-1$ and ends on the plane $z=1$ (see figure 3*b*).

The null value of the strength also allows vortex tubes to begin or end at points within the flow domain. Take for instance a fluid of infinite extent that is everywhere at rest except within the bulb $r<g(z)$, $-1<z<1$, where the velocity and vorticity fields are given, respectively, by

$$\boldsymbol{u} = f(r,z)g(z)\boldsymbol{\theta}, \tag{3.3}$$

$$\boldsymbol{\omega} = -\frac{\partial f(r,z)g(z)}{\partial z}\boldsymbol{r} + \left(\frac{\partial f(r,z)}{\partial r} + \frac{f(r,z)}{r}\right)g(z)\boldsymbol{k}, \tag{3.4}$$

with $g(z)=z^4-2z^2+1$ and $f(r,z)=[r-g(z)]^5-2[r-g(z)]^3+[r-g(z)]$. This is also a circular, axially symmetric flow, but now the radial extent of the motion diminishes with the distance from the plane $z=0$. The geometry of the vortex lines in a meridional plane is shown in figure 4(*a*). In this plane all vortex lines are closed except for two, which begin at $(r,z)=(0,-1)$ and end at $(r,z)=(0,-1)$. The vortex

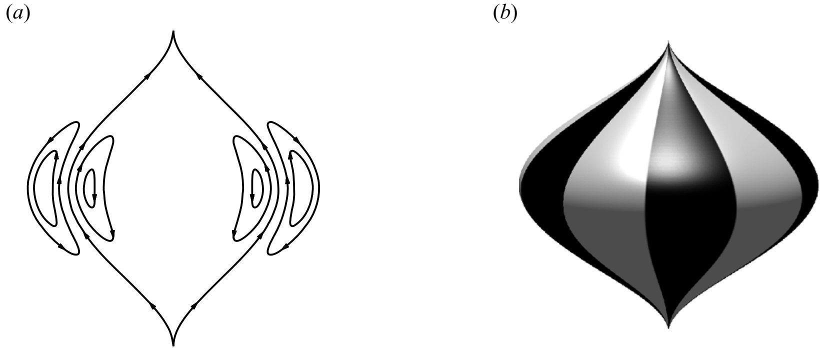

FIGURE 4. (*a*) The vorticity field given by equation (3.4) represented in a meridional plane. All vortex lines are closed, except for the separatrix vortex lines, which begin and end at two common points. (*b*) The vortex tube formed by all such separatrix vortex lines has zero strength and begins and ends at the points $(r, z) = (0, -1)$ and $(r, z) = (0, 1)$, respectively.

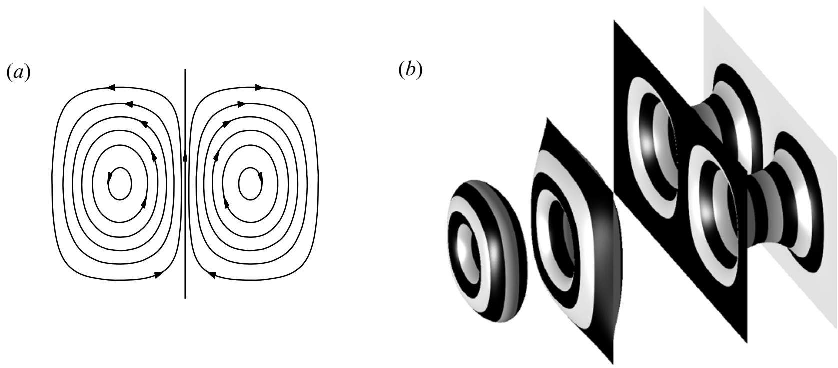

FIGURE 5. (*a*) The vorticity field given by equation (3.6) represented in the plane (y, z). All vortex lines in this plane are closed, except for the separatrix vortex line. (*b*) From left to right: a closed vortex tube, a vortex tube with an infinitesimal opening, and a vortex tube with a finite opening.

tube, which is the solid of revolution generated by these vortex lines, also begins and ends at these points (figure 4*b*).

The existence of equilibrium points of the vorticity field also has consequences for the topology of vortex tubes of finite strength. Consider a fluid that is everywhere at rest except within the region $-\infty < x < \infty$, $-1 < y < 1$, $-1 < z < 1$ where the velocity and vorticity fields are given, respectively, by

$$\boldsymbol{u} = f(y)g(z)\boldsymbol{i}, \tag{3.5}$$

$$\boldsymbol{\omega} = f(y)g'(z)\boldsymbol{j} - f'(y)g(z)\boldsymbol{k}, \tag{3.6}$$

with $f(y) = y^5 - 2y^3 + y$ and $g(z) = z^4 - 2z^2 + 1$. This is the parallel flow of two jets streaming in opposite directions along the x-axis. In a plane perpendicular to the flow the vortex lines have the geometry shown in figure 5(*a*); all of them are closed except for the separatrix vortex line that starts at $(y, z) = (0, -1)$ and ends at $(y, z) = (0, 1)$. Let us consider the vortex tubes formed by vortex lines that pass through a small circle located on the plane (x, y). If the circle does not intersect the x-axis the vortex tube contains only closed vortex lines; the tube is thus also closed

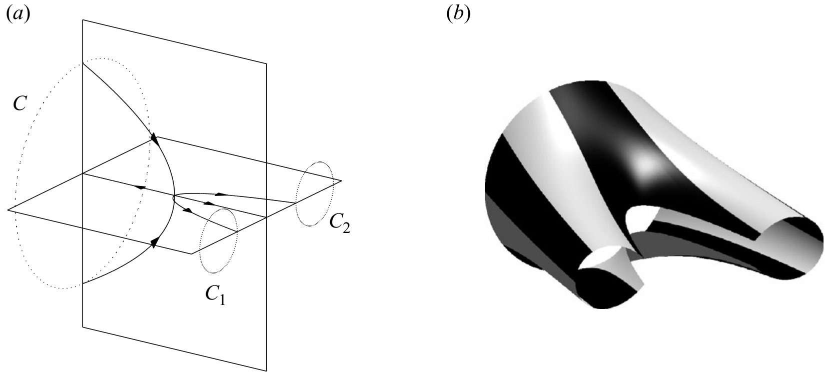

FIGURE 6. (*a*) Geometry of the vorticity field given by equation (3.8). All vortex lines lying on the plane (x, y) diverge from the equilibrium point at the origin, whereas two separatrix vortex lines lying on the plane (x, z) end at this point. (*b*) The vortex lines passing through contour C generate a vortex tube of finite strength that is torn apart at the origin; beyond this point there are no tubes but open surfaces because C cannot be mapped into C_1 and C_2.

and has the shape of a torus. If the circle touches the x-axis at exactly one point the vortex tube contains one separatrix vortex line; the tube has the shape of a squared doughnut but is not closed: it remains open by a slit of infinitesimal width. Finally, if the circle intersects the x-axis at two points the vortex tube contains two separatrix vortex lines and is torn apart at the extreme points of these lines. These shapes are illustrated in figure 5(*b*).

Our final example is a flow with the following velocity and vorticity distributions

$$\boldsymbol{u} = 2yz\boldsymbol{i} + z^3/3\boldsymbol{j} + (xy - y^3/3)\boldsymbol{k}, \tag{3.7}$$

$$\boldsymbol{\omega} = (x - y^2 - z^2)\boldsymbol{i} + y\boldsymbol{j} - 2z\boldsymbol{k}. \tag{3.8}$$

The geometry of this vorticity field can be summarized as follows (figure 6*a*). There is an equilibrium point at the origin, two separatrix vortex lines on the plane (x, z) begin at infinity and end at this point, and an infinite number of vortex lines on the plane (x, y) begin at the origin and end at infinity. Let C be a closed contour on the plane (y, z) that intersects transversally the two separatrix vortex lines (see figure 6*a*). The vortex lines that pass through C generate a vortex tube that is torn apart at the equilibrium point $(x, y, z) = (0, 0, 0)$. The two branches formed beyond the equilibrium point are not tubes but folded surfaces (figure 6*b*). The vortex lines shown in the figure were computed by integrating (2.1) for a short interval s, so that the gap at the fork and all along the two branches is clearly visible. If the interval of integration is increased the width of the gap decreases; and in the limit $s \to \infty$ the width becomes infinitely small. Yet the gap never closes because the branches lack two vortex lines which do not pass through C (they start at the null point and go in the positive x direction). The tearing up of the tube must happen because the flow $\boldsymbol{\Phi}_s$ generated by (2.1) preserves topology: a contour cannot be mapped into two contours; thus C cannot be mapped into C_1 and C_2 (figure 6*a*).

4. Conclusions

Vortex lines and tubes have been essential in describing and understanding the motion of fluids ever since Helmholtz (1858) published his seminal memoir on vortex motion. There, he established the fundamental dynamical laws of vorticity as well as a number of kinematic theorems. In particular, he proved that the vorticity flux is constant along a vortex filament and from this he concluded, incorrectly, that vortex filaments must close on themselves or extend to the boundary of the fluid. Soon afterwards the same deduction was made about vortex tubes (Thomson 1869) and vortex lines (Lamb 1879). This corollary is still widely accepted today even though its incorrectness was pointed out about one hundred years ago (Hadamard 1903). The analytic examples presented here display vortex lines and tubes which do not fit into those traditional categories. Thus in these simple flows we find dense vortex lines, which are bounded lines of infinite length that never close on themselves, and separatrix vortex lines, which are lines that begin or end within the fluid, at points where the vorticity is zero. For vortex tubes, the possibilities are wider because they may contain any combination of the various types of vortex lines. The following are representative examples. Vortex tubes formed by dense vortex lines (alone or in combination with closed lines) have infinite length, they self-intersect an infinite number of times but do not close on themselves. Vortex tubes formed by separatrix vortex lines and either open or closed lines are torn apart at the points where the vorticity is null; beyond these points the vortex surface becomes a set of stripes. Vortex tubes exclusively composed of separatrix vortex lines begin or finish at points or surfaces within the fluid; in this particular situation the vortex tube is also characterized as having zero strength.

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